Name KEY Date __________ Period ______

Worksheet 4.9—Derivatives of Exponential Functions

Show all work. No calculator unless otherwise stated.

For 1 – 8, Find $\frac{dy}{dx}$

1. $y = e^{(2x^2+2x)}$

$\frac{dy}{dx} = e^{(2x^2+2x)} \cdot (4x+2)$

$\boxed{\frac{dy}{dx} = (4x+2)e^{2x^2+2x}}$

2. $y = 6^{(2x)}$

$\frac{dy}{dx} = 6^{2x} (\ln 6) \cdot 2$

$\boxed{\frac{dy}{dx} = (2\ln 6)6^{2x}}$

or $\frac{dy}{dx} = (\ln 36) \cdot 6^{2x}$

3. $y = \sin^2 x + 2^{\sin x}$

$y = (\sin x)^2 + 2^{(\sin x)}$

$\frac{dy}{dx} = 2(\sin x)^1 \cos x + 2^{\sin x}(\ln 2)\cos x$

$\boxed{\frac{dy}{dx} = 2\sin x\cos x + (\ln 2)\cos x \cdot 2^{\sin x}}$

or $\frac{dy}{dx} = \underbrace{\sin 2x}_{\text{double angle ID}} + (\ln 2)\cos x \cdot 2^{\sin x}$

4. $y = xe^2 - e^{x^2}$

$y = (e^2)x - e^{(x^2)}$

$\frac{dy}{dx} = e^2 - e^{x^2} \cdot (2x)$

$\boxed{\frac{dy}{dx} = e^2 - 2xe^{x^2}}$

5. $y = \frac{e^x + e^{-x}}{4}$

$y = \frac{1}{4}(e^x + e^{-x})$

$\frac{dy}{dx} = \frac{1}{4}(e^x + e^{-x} \cdot (-1))$

$\boxed{\frac{dy}{dx} = \frac{1}{4}(e^x - e^{-x})}$

or $\frac{dy}{dx} = \frac{1}{4}e^{-x}(e^{2x} - 1)$

or $\frac{dy}{dx} = \frac{e^{2x}-1}{4e^x}$

6. $y = \left(2e^x - e^{-x}\right)^3$

$\frac{dy}{dx} = 3(2e^x - e^{-x})^2 \cdot (2e^x - e^{-x} \cdot (-1))$

$\boxed{\frac{dy}{dx} = 3(2e^x - e^{-x})^2(2e^x + e^{-x})}$

7. $y = 2^{-3/x}$

$y = 2^{-3x^{-1}}$

$\frac{dy}{dx} = 2^{-3/x}(\ln 2)(3x^{-2})$

$\frac{dy}{dx} = \frac{(3\ln 2)(2^{-3/x})}{x^2}$

or $\frac{dy}{dx} = \frac{3\ln 2}{x^2 \cdot 2^{3/x}}$

8. $5 = 3e^{xy} + x^2 y + xy^2$

(implicit differentiation)

$\frac{d}{dx}[5] = \frac{d}{dx}\left[3e^{(xy)} + (x^2)(y) + (x)(y^2)\right]$

$0 = 3e^{xy}\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + (2x)(y) + (x^2)\left(\frac{dy}{dx}\right) + (1)(y^2) + (x)\left(2y\frac{dy}{dx}\right)$

$0 = 3ye^{xy} + 3x\frac{dy}{dx}e^{xy} + 2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx}$

$-3ye^{xy} - 2xy - y^2 = \frac{dy}{dx}(3xe^{xy} + x^2 + 2xy)$

$\boxed{\frac{dy}{dx} = \frac{-3ye^{xy} - 2xy - y^2}{3xe^{xy} + x^2 + 2xy}}$

or $\frac{dy}{dx} = -\frac{3ye^{xy} + 2xy + y^2}{3xe^{xy} + x^2 + 2xy}$

9. Find the equation of the tangent and normal line to the graph of

(a) $y = xe^x - e^x$ at $(1,0)$

Need slope @ $(1,0)$, or $\left.\frac{dy}{dx}\right|_{(1,0)}$

$\frac{dy}{dx} = (1)(e^x) + (x)(e^x) - e^x$

$\frac{dy}{dx} = xe^x$

$\left.\frac{dy}{dx}\right|_{x=1} = 1 \cdot e^1 = \boxed{e}$

Tangent Line

$\boxed{y = 0 + e(x-1)}$

or $y = ex - e$

Normal Line

$\boxed{y = 0 - \frac{1}{e}(x-1)}$

or

$\boxed{y = -\frac{1}{e}x + \frac{1}{e}}$

(b) $xe^y + ye^x + 1 = 2e^x$ at $(0,1)$

(implicit differentiation)

$\frac{d}{dx}\left[xe^y + ye^x + 1\right] = \frac{d}{dx}\left[2e^x\right]$

$(1)(e^y) + (x)(e^y)\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)(e^x) + (y)(e^x) + 0 = 2e^x$

* since we only want the numeric slope, plug in (0,1) now, then solve for $\frac{dy}{dx}$.

At (0,1):

$e + (0) + \frac{dy}{dx} + 1 = 2$

$\left.\frac{dy}{dx}\right|_{(0,1)} = \boxed{1-e}$

Tangent Line

$\boxed{y = 1 + (1-e)(x-0)}$

Normal Line

$\boxed{y = 1 - \frac{1}{(1-e)}(x-0)}$

or $y = 1 + \frac{x}{e-1}$

10. Find $\frac{d^2y}{dx^2}$ for $y = 2\sin\left(4^{x^2}\right)$

$\frac{dy}{dx} = 2\cos\left(4^{x^2}\right)\cdot\left(4^{x^2}\right)(\ln 4)\cdot(2x)$

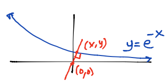
$\frac{dy}{dx} = \left[(4\ln 4)x\right]\left[4^{x^2}\right]\left[\cos\left(4^{x^2}\right)\right]$ *triple product

$\frac{d^2y}{dx^2} = \left[4\ln 4\right]\left[4^{x^2}\right]\left[\cos\left(4^{x^2}\right)\right] + \left[(4\ln 4)x\right]\left[4^{x^2}(\ln 4)\cdot(2x)\right]\left[\cos\left(4^{x^2}\right)\right] + \left[(4\ln 4)x\right]\left[4^{x^2}\right]\left[-\sin\left(4^{x^2}\right)\cdot\left(4^{x^2}\right)(\ln 4)\cdot(2x)\right]$

11. (Calculator OK) Find the point of the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin.

$y' = -e^{-x} = -\frac{1}{e^x}$

Normal slope $= e^x$



Slope of Normal Line

$\frac{y-0}{x-0} = \frac{-1}{y'(x)}$

$\frac{y}{x} = e^x$

$y = xe^x$

but $y = e^{-x}$, so

$e^{-x} = xe^x$

$xe^x - e^{-x} = 0$

from calculator: $\boxed{x = 0.426 = A}$ ALWAYS store non-exact values for later use, then label them on your paper with the same letter as which you stored them.

pt: $(0.426, e^{-0.426\ldots}) = \boxed{(0.426, 0.652)} = (A, B)$

12. (Calculator OK) Compare each of the following numbers with the number *e*. Is the number less than or greater than *e*?

$e = 2.718281828\ldots$ (on calculator, 2nd ÷)

(a) $\left(1+\frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

$= 2.718280469\ldots < e$

(b) $1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{720}+\frac{1}{5040}$

$= 2.709920635\ldots < e$

Multiple Choice

C 13. Find the value of $\lim_{x\to\infty}\left(\dfrac{2e^{2x}+5e^{-2x}}{e^{2x}-4e^{-2x}}\right)$

(A) $\dfrac{3}{5}$ (B) $-\dfrac{1}{2}$ (C) 2 (D) $-\dfrac{3}{5}$ (E) $\dfrac{1}{2}$ (F) -2

$\lim_{x\to\infty}\left(\dfrac{2e^{2x}+\frac{5}{e^{2x}}}{1e^{2x}-\frac{4}{e^{2x}}}\right)=\dfrac{2}{1}=\boxed{2}$

C 14. Determine $f'(x)$ when $f(x)=e^{\sqrt{3x+4}}$

(A) $f'(x)=\dfrac{3e^{\sqrt{3x+4}}}{\sqrt{3x+4}}$ (B) $f'(x)=\dfrac{3}{2}e^{\sqrt{3x+4}}\sqrt{3x+4}$ (C) $f'(x)=\dfrac{3e^{\sqrt{3x+4}}}{2\sqrt{3x+4}}$

(D) $f'(x)=3e^{\sqrt{3x+4}}$ (E) $f'(x)=\dfrac{e^{\sqrt{3x+4}}}{2\sqrt{3x+4}}$

$f(x)=e^{(3x+4)^{1/2}}$

$f'(x)=e^{\sqrt{3x+4}}\cdot\left(\frac{1}{2}\right)(3x+4)^{-1/2}\cdot(3)$

$f'(x)=\dfrac{3e^{\sqrt{3x+4}}}{2\sqrt{3x+4}}$

B 15. Find $\frac{dy}{dx}$ when $y=\cos\left(e^{x}\right)+e^{x}\sin\left(e^{x}\right)$

(A) $\frac{dy}{dx}=e^{2x}\sin\left(e^{x}\right)$ (B) $\frac{dy}{dx}=e^{2x}\cos\left(e^{x}\right)$ (C) $\frac{dy}{dx}=-e^{2x}\cos\left(e^{x}\right)$ (D) $\frac{dy}{dx}=e^{x}\cos\left(e^{x}\right)$

(E) $\frac{dy}{dx}=-e^{2x}\sin\left(e^{x}\right)$ (F) $\frac{dy}{dx}=e^{x}\sin\left(e^{x}\right)$ (G) $\frac{dy}{dx}=-e^{x}\sin\left(e^{x}\right)$ (H) $\frac{dy}{dx}=-e^{x}\cos\left(e^{x}\right)$

$\frac{dy}{dx}=\cancel{-\sin(e^x)\cdot e^x}+\cancel{e^x\sin(e^x)}+e^x\cos(e^x)\cdot e^x=e^{2x}\cos(e^x)$

E 16. Determine all values of r for which the function $y=e^{rx}$ satisfies the equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-8y=0$.

(A) $r=2,\ 4$ (B) $r=-3,\ 5$ (C) $r=-4,\ 2$ (D) $r=-4,\ -2$ (E) $r=-2,\ 4$

$y=e^{rx}$

$\frac{dy}{dx}=re^{rx}$

$\frac{d^2y}{dx^2}=r^2e^{rx}$

So $(r^2e^{rx})-2(re^{rx})-8(e^{rx})=0$

$e^{rx}[r^2-2r-8]=0$

$e^{rx}(r-4)(r+2)=0$

$\underbrace{e^{rx}=0}_{\text{No Solution}}$ or $r-4=0$ or $r+2=0$

$\boxed{r=4}$ $\boxed{r=-2}$

D 17. (Precal Review) If a, b are solutions to the equation $2e^x + 10e^{-x} = 9$, find the value of $a + b$.

(A) $e^{9/2}$ (B) 5 (C) $\ln 9 - \ln 2$ (D) $\ln 5$ (E) $\frac{9}{2}$

$2e^x + \frac{10}{e^x} - 9 = 0$

$\frac{2e^x}{1}\left(\frac{e^x}{e^x}\right) + \frac{10}{e^x} - \frac{9e^x}{e^x} = 0$

$\frac{2e^{2x} - 9e^x + 10}{e^x} = 0$

when $2e^{2x} - 9e^x + 10 = 0$

$(2e^x - 5)(e^x - 2) = 0$

$2e^x = 5$ or $e^x = 2$

$x = \ln\left(\frac{5}{2}\right)$ or $x = \ln 2$

So $x = \ln\left(\frac{5}{2}\right) = \boxed{\ln 5 - \ln 2 = a}$ and $x = \boxed{\ln 2 = b}$

$a+b = \ln 5 - \cancel{\ln 2} + \cancel{\ln 2}$

$= \boxed{\ln 5}$

D 18. If f is the function defined by $f(x) = e^{2x} + 6e^{-2x}$, find the value of $f'(\ln 2)$.

(A) 6 (B) $\frac{9}{2}$ (C) $\frac{11}{2}$ (D) 5 (E) $\frac{13}{2}$

$f'(x) = 2e^{2x} - 12e^{-2x}$

$f'(x) = 2e^{2x} - \frac{12}{e^{2x}}$

$f'(\ln 2) = 2e^{2\ln 2} - \frac{12}{e^{2\ln 2}}$

$f'(\ln 2) = 2e^{\ln 2^2} - \frac{12}{e^{\ln 2^2}}$ $\quad * \; e^{\ln u} = u$

$f'(\ln 2) = 2(2^2) - \frac{12}{2^2}$

$f'(\ln 2) = 8 - 3 = \boxed{5}$

D 19. If $f(x) = x^3 e^{2x}$, on what interval(s) is $f'(x) \ge 0$?

(A) $(-\infty, 0] \cup \left[\frac{3}{2}, \infty\right)$ (B) $\left(-\infty, -\frac{3}{2}\right]$ (C) $\left(-\infty, -\frac{3}{2}\right] \cup [0, \infty)$ (D) $\left[-\frac{3}{2}, \infty\right)$ (E) $\left(-\infty, \frac{3}{2}\right]$

$f'(x) = (3x^2)(e^{2x}) + (x^3)(e^{2x} \cdot 2)$

$f'(x) = e^{2x}(3x^2 + 2x^3) = 0$

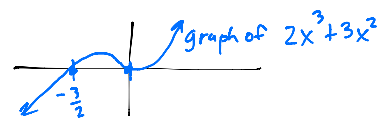
$e^{2x} = 0$
No Solution
$e^{2x} > 0 \; \forall x \in \mathbb{R}$

$x^2(3 + 2x) = 0$

$x^2 = 0$ $\quad x = -\frac{3}{2}$

$x = 0$ (M2) graphical bounce $\quad$ (M1) graphical cross

Since $e^{2x} \ge 0 \; \forall x \in \mathbb{R}$,
$f'(x) \ge 0$ when $3x^2 + 2x^3 \ge 0$



this graph is $\ge 0 \; \forall x \ge -\frac{3}{2}$

So $f'(x) \ge \; \forall x \in \left[-\frac{3}{2}, \infty\right)$